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SOME MODULAR IDENTITIES OF RAMANUJAN USEFUL IN APPROXIMATING π

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ABSTRACT. We show how various modular identities due to Ramanujan may be used to produce simple high order approximations to π. Various specializations are considered and the Gaussian arithmetic geometric mean formula for π is rederived as a consequence.

1. In the second part of Ramanujan's 1914 paper *Modular equations and approximations to* π [**9**], the author lists some remarkable modular identities which he uses to give algebraic approximations to π (such as

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right)$$

which gives π to 11 digits). While considerable attention has been paid to the prodigious singular value calculations which comprise the first half of the paper ([**8**, **12**] and references therein), less attention seems to have been given to the subtle identities given in Table III of [**9**]. In this note we show how these identities lead to explicit nth order approximations to π.

We begin by sketching the genesis of the approximation and then give explicit specializations for various integers. We finish by rederiving the Gaussian arithmetic geometric mean (AGM) identity for π which forms the basis for the recent high precision calculation of π (16 million decimal digits) by Tamura and Kanada ([**11**] and private communications). For more information on the AGM the reader is referred to [**3**].

2. We take for granted the basic identities of elliptic and theta function theory as available in Whittaker and Watson [**13**], Cayley [**7**] and Bellman [**1**]. A more explicit treatment will be forthcoming in [**5**].

Ramanujan proceeds as follows. Let n be a positive integer and let $\Phi_n(k,l)$ denote the nth order *modular equation* which is algebraic in k,l and polynomial in $u := k^{1/4}$, $v := l^{1/4}$. Suppose that $\Phi_n(k,l) = 0$ with $0 < l < k$ and let $K := K(k)$, $L := K(l)$ denote the (complete) *elliptic integrals* (of the first kind) with moduli k and l, respectively. As usual let $k' := \sqrt{1-k^2}$ denote the *conjugate modulus* and $K' := K'(k) := K(k')$. Then

$$(1) \qquad \text{(i)} \quad n\frac{K'}{K} = \frac{L'}{L} \quad \text{and} \quad \text{(ii)} \quad L = m_n K$$

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where $m_n := m_n(k,l)$ is the associated *multiplier* which is also algebraic in k and l. Indeed Jacobi's differential equation

$$n\frac{dk}{dl} = \frac{kk'^2}{ll'^2}\left(\frac{K}{L}\right)^2 = \frac{kk'^2}{ll'^2}m_n^{-2} \tag{2}$$

shows immediately the algebraic nature of m_n, given that for Φ_n. In terms of the *nome* q one has

$$q := e^{-\pi K'/K} \quad \text{and} \quad q^n = e^{-\pi L'/L} \tag{3}$$

and the standard product relationship

$$\frac{q^{1/12}\prod_{k=1}^{\infty}(1-q^{2k})}{q^{n/12}\prod_{k=1}^{\infty}(1-q^{2kn})} = \left(\frac{kk'}{ll'}\right)^{1/6}\sqrt{\frac{K}{L}}. \tag{4}$$

Logarithmic differentiation of (4) combined with application of (2) and $qdk/dq = 2kk'^2K^2/\pi^2$ produce

$$nP(q^n) - P(q) = (4KL/\pi^2)R_n(k,l), \tag{5}$$

where

$$P(q) := 1 - 24\sum_{m=1}^{\infty}\frac{mq^{2m}}{1-q^{2m}}$$

and R_n is an algebraic function of k and l. The algebraic nature of R_n follows from (2). Using calculations similar to those in §5 of [**4**] one can give a reasonably simple general formula for R_n. No details are given by Ramanujan who, however, then lists elegant specializations of R_n for various different n. (Specifically $2, 3, 5, 7, 11, 15, 17, 19, 23, 31, 35$. The equation given of degree 4 appears flawed.)

Thus one has the following tables in which R_n is taken from Table III of [**9**], and the modular information is available for the most part in [**7** or **5**]. Indeed Cayley gives $\Phi_n(u,v)$ for all odd n up to 20, and $m_n(u,v)$ is then computable from (2). As before $u := k^{1/4}$, $v := l^{1/4}$.

(A) *Modular equations*

$$\begin{aligned}
\Phi_2(k,l) &= (1+k')l - (1-k') = 0,\\
\Phi_3(k,l) &= (kl)^{1/2} + (k'l')^{1/2} - 1 = 0 \quad \textit{or}\\
\Phi_3(u,v) &= u^4 - v^4 + 2(uv)^3 - 2uv = 0,\\
\Phi_5(k,l) &= (k^{1/2} - l^{1/2})^3 - 4(kl)^{1/4}(k'l') = 0 \quad \textit{or}\\
\Phi_5(u,v) &= u^6 - v^6 + 5u^2v^2(u^2 - v^2) - 4uv(1 - u^4v^4) = 0,\\
\Phi_7(k,l) &= (kl)^{1/4} + (k'l')^{1/4} - 1 = 0 \quad \textit{or}\\
\Phi_7(u,v) &= (1-u^8)(1-v^8) - (1-uv)^8 = 0,\\
\Phi_{23}(k,l) &= (kl)^{1/4} + (k'l')^{1/4} + 2^{2/3}(klk'l')^{1/2} - 1 = 0.
\end{aligned}$$

(B) *Multipliers*

$$m_2(k,l) = \frac{1+k'}{2} = \frac{1}{1+l},$$
$$m_3(u,v) = \frac{u}{u+2v^3} = \frac{2u^3-v}{3v},$$
$$m_5(u,v) = \frac{v+u^5}{5(v+uv^4)} = \frac{u-vu^4}{u-v^5},$$
$$m_7(u,v) = \frac{u(1-uv)(1-uv+(uv)^2)}{u-v^7} = \frac{u^7-v}{7v(1-uv)(1-uv+(uv)^2)},$$
$$m_{13}(k,l) = \left(\frac{l}{k}\right)^{1/2} + \left(\frac{l'}{k'}\right)^{1/2} - \left(\frac{ll'}{kk'}\right)^{1/2} - 4\left(\frac{ll'}{kk'}\right)^{1/3} = \frac{13}{m_{13}(l,k)}.$$

(C) *Ramanujan identities*

$$R_2(k,l) = k'+l,$$
$$R_3(k,l) = 1+kl+k'l',$$
$$R_5(k,l) = (3+kl+k'l')\sqrt{\frac{1+kl+k'l'}{2}},$$
$$R_7(k,l) = 3(1+kl+k'l'),$$
$$R_{15}(k,l) = [1+(kl)^{1/4}+(k'l')^{1/4}]^4 - (1+kl+k'l'),$$
$$R_{19}(k,l) = 6[(1+kl+k'l')+(kl)^{1/2}+(k'l')^{1/2}-(kk'll')^{1/2}],$$
$$\begin{aligned} R_{23}(k,l) &= 11(1+kl+k'l') \\ &\quad - 16(4kk'll')^{1/6}(1+(kl)^{1/2}+(k'l')^{1/2}) - 20(4kk'll')^{1/3}, \end{aligned}$$
$$\begin{aligned} R_{35}(k,l) &= 2[(kl)^{1/2}+(k'l')^{1/2}-(kk'll')^{1/2}] \\ &\quad + 4(kk'll')^{-1/6}[1-(kl)^{1/2}-(k'l')^{1/2}]^3. \end{aligned}$$

Moreover, if one substitutes $\overline{q} := e^{-\pi/\sqrt{n}}$ in (4) before differentiating logarithmically one derives

$$nP(e^{-\pi\sqrt{n}}) + P(e^{-\pi/\sqrt{n}}) = 6\sqrt{n}/\pi, \tag{6}$$

and setting $n=1$ shows

$$P(e^{-\pi}) = 3/\pi. \tag{7}$$

The interested reader is directed to [**2**] for much recent information on identities like (6).

We now diverge from Ramanujan whose purpose was to provide explicit algebraic approximations to π, while we are concerned with reduced complexity iterative approximations for π. Let $k_0 := 1/\sqrt{2}$ and iteratively solve $\Phi_n(k_i, k_{i+1})$ for k_{i+1}, $1 \le i < \infty$. If we set $K_i := K(k_i)$ we have

$$n^{N+1}P\left(e^{-\pi n^{N+1}}\right) - \frac{3}{\pi} = \sum_{i=0}^{N} n^i \frac{4K_iK_{i+1}}{\pi^2} R_n(k_i, k_{i+1}), \tag{8}$$

as follows on summing (5) and using (7). Moreover,

$$0 \le 1 - P(e^{-\pi n^N}) \le 25e^{-2\pi n^N},$$

and asymptotically one can replace "25" by "24". Thus,

$$(9) \qquad 0 \leq \left[n^{N+1} - \sum_{i=0}^{N} n^i \frac{4K_i K_{i+1}}{\pi^2} R_n(k_i, k_{i+1}) \right] - \frac{3}{\pi} \leq 100 n^{N+1} e^{-2\pi n^{N+1}}.$$

Since

$$n^{N+1} = 1 + (n-1) \sum_{i=0}^{N} n^i$$

this yields

$$(10) \qquad \pi = 3 \left(1 - \sum_{i=0}^{\infty} n^i \left[\frac{4K_i K_{i+1}}{\pi^2} R_n(k_i, k_{i+1}) - (n-1) \right] \right)^{-1}$$

in which the convergence is nth order. We now observe that $K_0 = K(1/\sqrt{2}) = \pi/[2M(1, 1/\sqrt{2})]$ where $M(1, 1/\sqrt{2}) =: M_0$ is the Gaussian arithmetic geometric mean of 1 and $1/\sqrt{2}$ [**3**]. Then if we set $a_i := (2K_i/\pi)M_0$ we have

$$(11) \qquad \pi = 3 \left(1 - \sum_{i=0}^{\infty} n^i \left[\frac{a_i a_{i+1}}{M_0^2} R_n(k_i, k_{i+1}) - (n-1) \right] \right)^{-1}$$

where $k_0 := 1/\sqrt{2}$, $a_0 := 1$,

(i) $\Phi_n(k_i, k_{i+1}) = 0$, $k_{i+1} \in (0, k_i)$,

(ii) $a_{i+1} := m_n(k_i, k_{i+1}) a_i$, and

(iii) $M_0 = \lim_{i \to \infty} a_i$.

Here (ii) is a consequence of (1)(ii), and (iii) follows since $\lim_{i\to\infty} K_i = K(0) = \pi/2$. Thus (11) and (12) give algebraic series whose sum is π and whose convergence is order n. While (11) leads to more elegant formulae for π, we may also write

$$(12) \qquad \begin{aligned} 0 &\leq \pi - 3 \left(1 - \sum_{i=0}^{N} n^i \left[\frac{a_i a_{i+1}}{a_{N+1}^2} R_n(k_i, k_{i+1}) - (n-1) \right] \right)^{-1} \\ &\leq 100 n^{N+1} e^{-2\pi n^{N+1}}, \end{aligned}$$

since $0 \leq a_{N+1} - M_0 \leq 4e^{-\pi n^{N+1}}$. This shows that one can compute π from one sequence of moduli.

Various adaptations of (12) are possible in which one replaces $k_0 := 1/\sqrt{2}$ by other singular values. Indeed, (5) and (6) combine to give

$$(13) \qquad \sqrt{n} P(e^{-\pi\sqrt{n}}) = \frac{3}{\pi} + \frac{2K^2(\overline{k}_n)}{\pi^2} R_n(\overline{k}_n, \overline{k}_n')$$

where $\overline{k}_n$ solves $K'(\overline{k}_n) = \sqrt{n} K(\overline{k}_n)$. This is also the genesis of Ramanujan's explicit approximations. For $n > 1$, however, the formulae become more complicated, and involve $M(1, \overline{k}_n)$.

3. We now combine the information tabulated in §2 with (11) and (12) to produce the following approximations.

Quadratic. Using the AGM form of Φ_2 (as in [**4**]) leads to

(14) Let (i) $a_0 := 1,\ b_0 := 1/\sqrt{2}$,
(ii) $a_{n+1} := (a_n + b_n)/2,\ b_{n+1} := \sqrt{a_n b_n}$.
Then $M_0 = \lim_{n\to\infty} a_n$ and
(iii)

$$\pi = 3\left(1 - \sum_{n=0}^{\infty} 2^n \left[\left(\frac{a_n^2 + b_n^2}{2M_0^2}\right) - 1\right]\right)^{-1}.$$

Cubic. Using the u, v form of Φ_3 leads to

(15) Let (i) $a_0 := 1,\ v_0 := 2^{-1/8}$,
(ii) $a_{n+1} := a_n v_n/(v_n + 2v_{n+1}^3)$ where $\Phi_3(v_n, v_{n+1}) = 0$ and $0 < v_{n+1} < v_n$.
Then $M_0 = \lim_{n\to\infty} a_n$ and
(iii)

$$\pi = 3\left(1 - 2\sum_{n=0}^{\infty} 3^n \left[\frac{a_n a_{n+1}}{M_0^2}(1 + (v_n v_{n+1})^4 - (v_n v_{n+1})^2) - 1\right]\right)^{-1}.$$

Quartic. Combining two steps of the quadratic iteration yields

(16) Let (i) $x_0 := 1,\ y_0 := 2^{-1/4}$,
(ii) $x_{n+1} := (x_n + y_n)/2$ and $y_{n+1} := ((x_n y_n^3 + y_n x_n^3)/2)^{1/4}$.
Then $M_0 = \lim_{n\to\infty} x_n^2$ and
(iii)

$$\pi = 3\left(1 - 3\sum_{n=0}^{\infty} 4^n \left[\left(\frac{x_n^2 + y_n^2}{2M_0}\right)^2 - 1\right]\right)^{-1}.$$

Septic. Using the u, v form of Φ_7 leads to

(17) Let (i) $a_0 := 1,\ v_0 := 2^{-1/8}$,
(ii) $a_{n+1} := a_n v_n(1 - v_n v_{n+1})(1 - v_n v_{n+1} + (v_n v_{n+1})^2)/(v_n - v_{n+1}^7)$
where $\Phi_7(v_n, v_{n+1}) = 0$ and $0 < v_{n+1} < v_n$.
Then $M_0 := \lim_{n\to\infty} a_n$ and
(iii)

$$\pi = 3\left(1 - 3\sum_{n=0}^{\infty} 7^n \left[\frac{a_n a_{n+1}}{M_0^2}(1 + (v_n v_{n+1})^4 + (1 - v_n v_{n+1})^4) - 2\right]\right)^{-1}.$$

The interested reader will be able to produce similar approximations based on 5, 15, 35 and slightly less explicitly on 23 or other integers. In each case the error is given by (13). For example, the septic algorithm ((17) with a_{N+1} replacing M_0 and N replacing ∞) gives 16, 130, and 932 digits of π when run with $N = 3$.

The estimate given in (12) predicts 15, 129, and 931 digits respectively. Using $N = 9$, (17) will produce $7 \cdot 7 \times 10^8$ digits of π. For discussion of the computational complexity of such iterations the reader is referred to [**3** or **6**].

4. We now show that, at least, in the quadratic and cubic cases, one can cleanly remove the M_0^2 in (11). This results in a rederivation of the Gaussian identity for π. The argument relies on the following Proposition:

PROPOSITION. *If M_0, a_n, b_n are given by* (14) *then*

$$\frac{3}{2} + \sum_{n=0}^{N} 2^n(2b_n^2 - a_n^2) = 2^{N+1}M_0^2 + O(e^{-\pi 2^N}). \tag{18}$$

PROOF. Let

$$S(q) := 1 - 24\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1+q^{2n+1}}.$$

Then in theta function terms [**5**, **7**],

$$2\theta_4^4(q) - \theta_3^4(q) = S(q),$$

and, with $\overline{q} := e^{-\pi}$,

$$a_n^2 = M_0^2\theta_3^4\left(\overline{q}^{2^n}\right); \qquad b_n^2 = M_0^2\theta_4^4\left(\overline{q}^{2^n}\right).$$

Then

$$\begin{aligned}\sum_{n=0}^{N} 2^n\left[\frac{2b_n^2 - a_n^2}{M_0^2}\right] &= 2^{N+1} - 1 - 24\sum_{m=0}^{N}\sum_{n=0}^{\infty} \frac{2^m(2n+1)\overline{q}^{2^m(2n+1)}}{1+\overline{q}^{2^m(2n+1)}} \\ &= 2^{N+1} - 1 - 24\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{2^m(2n+1)\overline{q}^{2^m(2n+1)}}{1+\overline{q}^{2^m(2n+1)}} + O(\overline{q}^{2^N}) \\ &= 2^{N+1} - \left(1 + 24\sum_{n=0}^{\infty}\frac{n\overline{q}^n}{1+\overline{q}^n}\right) + O(q^{-2^N}).\end{aligned}$$

But

$$1 + 24\sum_{n=0}^{\infty}\frac{nq^n}{1+q^n} = \theta_2^4(q) + \theta_3^4(q).$$

Thus as $\theta_3^4(\overline{q}) + \theta_2^4(\overline{q}) = \frac{3}{2}\theta_3^4(\overline{q}) = 3/(2M_0^2)$ we have established (18). □

If we combine the information in (14) and in (18) we deduce that for large N

$$\begin{aligned}\frac{3}{\pi} &\sim 2^{N+1} - \sum_{n=0}^{N} 2^n\left(\frac{a_n^2 + b_n^2}{2M_0^2}\right) \quad \text{(from (14))} \\ &\sim \frac{3}{2M_0^2} - 3\sum_{n=0}^{N} 2^n\left(\frac{a_n^2 - b_n^2}{2M_0^2}\right) \quad \text{(from (18))}.\end{aligned}$$

Thus

$$\pi = \frac{2M_0^2}{1 - \sum_{n=0}^{\infty} 2^n(a_n^2 - b_n^2)} \tag{19}$$

which is the Gaussian identity for π rediscovered independently by Brent and Salamin in the 1970s and suggested as a complexity reducing basis for computation of π [**6**, **10**].

One can also establish that

$$\pi = \frac{2M_0^2}{1 - 4\sum_{n=0}^{\infty} 3^n a_n a_{n+1} (v_n v_{n+1})^2}, \tag{20}$$

where a_n and v_n are generated by (15). This can be done from a proposition analogous to the above, but is more easily seen from remanipulation of the cubic iteration given in [**4**].

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